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Umbralized Umbral Operators

S. A. Joni*

1. Introduction

In this paper we shall present a theory which arises from the "umbralization" of umbral operators. The English translation of the Latin word "umbra" is "shadow". Let $\{a_n\}_{n\geq 0}$ denote a sequence of numbers or polynomials. Umbral operations on the given sequence $\{a_n\}_{n\geq 0}$ were originally understood as formal manipulations in which, first, one "pretends" that the sequence in question is the power sequence $\{a^n\}_{n\geq 0}$, then one performs the desired operations, and finally one lowers all the exponents to their "shadows" (or subscripts), thereby obtaining valid identities for the sequence $\{a_n\}_{n\geq 0}$. One such umbral operation is the *umbral composition* of polynomial sequences. Let $\{a_n(x)\}_{n\geq 0}$ and $\{b_n(x)\}_{n\geq 0}$ denote sequences of polynomials, and suppose

$$a_n(x) = \sum_{k=0}^{n} a_{n,k}x^k. \tag{1.1}$$

The *umbral composition* $\{a_n(\mathbf{b})\}_{n\geq 0}$ of these two sequences is defined by setting, for all $n \geq 0$,

$$a_n(\mathbf{b}) = \sum_{k=0}^{n} a_{n,k}b_k(x). \tag{1.2}$$

We may think of this composition as first "pretending" that the sequence $\{b_k(x)\}_{k\geq 0}$ is the sequence $\{b(x)^k\}_{k\geq 0}$, then performing the usual composition

$$a_n(b(x)) = \sum_{k=0}^{n} a_{n,k}b(x)^k,$$

and finally lowering the exponents of $b(x)^k$ to the "shadows" $b_k(x)$, obtaining formula (1.2).

Before proceeding further, we need to establish some notational conventions and terminology. Let K denote a field of characteristic zero, $K[x]$ denote the algebra of polynomials in the variable x with coefficients in K, and $K[[u]]$ denote the algebra of formal power series in the variable u with coefficients in K. If $f(u) = \sum_{\nu=0}^{\infty} f_\nu u^\nu$, then we denote the *coefficient of* u^n *in* $f(u)$ by the symbol $f(u)|_{u^n}$. That is, for all $n \geq 0$,

$$f(u)|_{u^n} = f_n.$$

We shall say that the *index* of a formal power series $f(u)$ equals m if m is the smallest integer such that $f(u)|_{u^m}$ is non-zero. A formula power series $S(u)$ is said to be *multiplicatively invertible* if there exists a formal power series $R(u)$ such that $S(u)R(u) = 1$; $S(u)$ is multiplicatively invertible if and only if the index of $S(u)$ equals zero. A formal power series $f(u)$ is said to be *compositionally invertible* if there exists a formal power series $F(u)$ such that

$$f(F(u)) = F(f(u)) = u.$$

It is well known [2] that $f(u)$ is compositionally invertible if and only if the index of $f(u)$ equals one.

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A *polynomial sequence* is a sequence $\{p_n(x)\}$ such that, for all $n \geq 0$, $p_n(x)$ is a polynomial of degree precisely equal to n. A polynomial sequence $\{p_n(x)\}_{n\geq 0}$ is said to be of *binomial type* [7] if for all x, y, and all integers $n \geq 0$,

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y). \tag{1.3}$$

An *umbral operator* U [3] is a K-linear map of $K[x]$ to itself such that if we set, for all $n \geq 0$,

$$Ux^n = p_n(x),$$

then the sequence $\{p_n(x)\}_{n\geq 0}$ is a polynomial sequence of binomial type. We shall denote the formal *differentiation* operator by D, and the formal *evaluation at zero* operator by L_0. More precisely we set, for all $n \geq 0$,

$$Dx^n = nx^{n-1} \tag{1.4}$$

and

$$L_0 x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

Given a compositionally invertible formal power series $f(u)$, we set

$$U_f = \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu!} L_0 f(D)^\nu. \tag{1.6}$$

Garsia, in [3], shows that, corresponding to each umbral operator U, there exists a unique compositionally invertible formal power series $f(u)$ such that $U = U_f$, and it is this characterization of umbral operators that we shall presently umbralize. Indeed, umbralized umbral operators arise when we "lower" the exponents of $f(D)$ in Formula (1.6).

A sequence of formal power series $\mathbf{f} = \{f_n(u)\}_{n\geq 0}$ is said to be a *delta sequence* if

$$\operatorname{index} f_0(u) = 0 \tag{1.7}$$

and for all $n \geq 1$,

$$\operatorname{index} f_n(u) = 1. \tag{1.8}$$

A sequence $\mathbf{f}$ is said to be a *proper sequence* if it is a delta sequence and in addition satisfies

$$f_0(0) = 1, \tag{1.9}$$

and for all $n \geq 1$,

$$f'_n(0) = 1, \tag{1.10}$$

where $f'_n(u)$ denotes the formal derivative of the series $f_n(u)$. Finally, if $\mathbf{f}$ is a delta sequence and if $f_0(u) \equiv 1$, we shall say that $\mathbf{f}$ is a *normalized sequence*.

Let $\mathbf{f} = \{f_n(u)\}_{n\geq 0}$ be a delta sequence. The *umbralized umbral operator associated to* $\mathbf{f}$, $U_{\mathbf{f}}$, is defined by setting

$$U_{\mathbf{f}} = \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu!} L_0 f_0(D) f_1(D) \cdots f_\nu(D). \tag{1.11}$$

A polynomial sequence $\{p_n(x)\}_{n\geq 0}$ is said to be the *umbralized polynomial sequence relative to* $\mathbf{f}$ if $\mathbf{f}$ is a delta sequence, and if, for all $n \geq 0$,

$$p_n(x) = U_{\mathbf{f}} x^n.$$

The "umbralization" in Formula (1.11) may appear, at first glance, to involve more than a mere lowering of exponents to their shadows. However, note that an operator U is an umbralized operator if and only if there exists a sequence $\mathbf{g}=\{g_n(u)\}_{n\geq 0}$ such that, for all $n\geq 0$,

$$\text{index } g_n(u)=n, \tag{1.12}$$

and

$$U=\sum_{\nu=0}^{\infty}\frac{n^{\nu}}{\nu!}L_0 g_\nu(D). \tag{1.13}$$

This is easily seen since if $\mathbf{f}$ is a delta sequence, $U=U_{\mathbf{f}}$, and if we set, for all $n\geq 0$,

$$g_n(u)=f_0(u)f_1(u)\cdots f_n(u),$$

then U satisfies (1.12) and (1.13). Conversely, if (1.12) and (1.13) hold and if we set $f_0(u)=g_0(u)$ and, for all $n\geq 1$,

$$f_n(u)=g_n(u)/g_{n-1}(u),$$

then clearly $\mathbf{f}=\{f_n(u)\}_{n\geq 0}$ is a delta sequence and $U=U_{\mathbf{f}}$.

Let r denote a finite positive integer. A sequence of formal power series $\mathbf{f}=\{f_n(u)\}_{n\geq 0}$ is said to be an *r-sequence* if r is the smallest integer such that, for all $k\geq 1$,

$$f_k(u)=f_{k+r}(u).$$

Note that an r-sequence is completely determined by the finite sequence of formal power series $\{f_0(u), f_1(u), \ldots, f_r(u)\}$, and we shall identify the infinite r-sequence $\mathbf{f}$ with this finite sequence. If f is a delta r-sequence, we shall call its corresponding umbralized umbral operator $U_{\mathbf{f}}$ (see (1.11)) an *r-umbralized operator.* Moreover, a polynomial sequence $\{p_n(x)\}_{n\geq 0}$ is said to be the *r-umbralized polynomial sequence relative to* $\mathbf{f}$ if $\mathbf{f}$ is a delta r-sequence and if, for all $n\geq 0$,

$$p_n(x)=U_{\mathbf{f}}x^n.$$

It follows from the work of Garsia [3] that polynomial sequences of binomial type are umbralized polynomial sequences relative to a normalized 1-sequence, and that the closely related Sheffer sequences [9] are umbralized sequences relative to arbitrary delta 1-sequences.

In the following sections we shall be primarily concerned with the presentation of a generating function characterization of r-umbralized polynomial sequence and the study of their convolution identities. In addition we shall characterize the "C-equivalence" classes [5] of r-umbralized polynomial sequences and discuss other related results.

II. Multisections of the Exponential Series

One of the tools that we shall need in our study of r-umbralized polynomial sequences is the notion of the *multisections* of a series. Let $f(u)=\sum_{\nu=0}^{\infty} f_\nu u^\nu$ be a given formal power series. Following the notation of Riordan [8], we define, for $0\leq k\leq r-1$, the *k-mod*(r) *multisection of* $f(u)$ to be the series

$$f_k(u;r)=\sum_{\nu=0}^{\infty} f_{k+\nu r}x^{k+\nu r}. \tag{2.1}$$

Multisectioned series have been previously employed in combinatorial studies. In particular, Riordan [8] uses them in the study of certain sequences of combinatorial numbers.

For our purposes here, we shall be mostly concerned with multisections of the exponential series.

Throughout this paper we shall let r denote a given, positive integer, and ω denote a primitive rth root of unity.

PROPOSITION 2.1. *Let $f(u)=\sum_{\nu=0}^{\infty} f_\nu u^\nu$ be a given formal power series. Then the k-mod(r) multisection of $f(u)$ is given by*

$$f_k(u;r)=\frac{1}{r}\sum_{j=0}^{r-1}\omega^{-kj}f(\omega^j u). \tag{2.2}$$

PROOF. The proof of this proposition is immediate once we note that the coefficient of u^ν in the right-hand side of (2.2) is

$$\frac{1}{r}f_\nu\sum_{j=0}^{r-1}\omega^{(\nu-k)j}=\begin{cases} f_\nu & \text{if } \nu\equiv k \bmod(r) \\ 0 & \text{otherwise.}\end{cases}$$

We shall denote, for $0\leqslant k\leqslant r-1$ the k-mod(r) multisections of $\exp(u)$ by $\exp_k(u;r)$. For notational convenience, we extend this definition periodically to all integers. Thus, if l is a given integer and $l\equiv k \bmod(r)$ then $\exp_l(u;r)=\exp_k(u;r)$. In Section V we shall study convolution formulas for r-umbralized polynomial sequences. The convolution Formula (1.1) for binomial type sequences follows from the elementary fact that $\exp(a+b)=\exp(a)\exp(b)$. Similarly, our convolution formulas for r-umbralized polynomial sequences follow from the addition formulas for the multisections of $\exp(u)$. These formulas are the content of the following proposition.

PROPOSITION 2.2. *For all a and b,*

$$\exp_k(a+b;r)=\sum_{\nu=0}^{r-1}\exp_\nu(a;r)\exp_{k-\nu}(b;r). \tag{2.3}$$

PROOF. Using Proposition 2.1 we have that

$$\sum_{\nu=0}^{r-1}\exp_\nu(a;r)\exp_{k-\nu}(b;r)=\frac{1}{r^2}\sum_{j=0}^{r-1}\sum_{l=0}^{r-1}\omega^{-kl}\exp(\omega^j a+\omega^l b)\sum_{\nu=0}^{r-1}\omega^{\nu(l-j)}. \tag{2.4}$$

Now, for fixed j and l,

$$\sum_{\nu=0}^{r-1}\omega^{\nu(l-j)}=\begin{cases} r & \text{if } j=l \\ 0 & \text{otherwise.}\end{cases} \tag{2.5}$$

Thus, substituting (2.5) into (2.4) gives

$$\begin{aligned}\sum_{\nu=0}^{r-1}\exp_\nu(a;r)\exp_{k-\nu}(b;r)&=\frac{1}{r}\sum_{j=0}^{r-1}\omega^{-jk}\exp(\omega^j(a+b))\\ &=\exp_k(a+b;r).\end{aligned} \tag{2.6}$$

III. UMBRALIZED SEQUENCES

Let us recall from Section I that a polynomial sequence $\{p_n(x)\}$ is the *umbralized polynomial sequence relative to* $\mathfrak{f}$ if $\mathfrak{f}$ is a delta sequence and, for all $n\geqslant 0$,

$$p_n(x)=U_{\mathfrak{f}}x^n, \tag{3.1}$$

where $U_{\mathfrak{f}}$ is as defined in Formula (1.11).

THEOREM 3.1. *Let $\mathfrak{f}$ denote a delta sequence. Then $\{p_n(x)\}$ is the umbralized sequence relative to $\mathfrak{f}$ if and only if*

$$\sum_{n=0}^{\infty} p_n(x)\frac{u^n}{n!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} f_0(u)f_1(u)\cdots f_k(u). \tag{3.2}$$

PROOF. Formula (3.1) is equivalent to the fact that for all $n, k \geqslant 0$,

$$\begin{aligned} p_n(x)|_{x^k/k!} &= L_0 f_0(D)\cdots f_k(D)x^n \\ &= f_0(u)\cdots f_k(u)|_{u^n/n!}. \end{aligned} \tag{3.3}$$

Therefore, multiplying both sides of (3.3) by $(u^n/n!)(x^k/k!)$ and summing over all $n, k \geqslant 0$ gives

$$\begin{aligned} \sum_{n=0}^{\infty} p_n(x)\frac{u^n}{n!} &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} (f_0(u)\cdots f_k(u)|_{u^n/n!})\frac{u^n}{n!}\frac{x^k}{k!} \\ &= \sum_{k=0}^{\infty}\frac{x^k}{k!} f_0(u)\cdots f_k(u). \end{aligned}$$

Conversely, if (3.2) holds, then, for any $n, k \geqslant 0$, taking the coefficient of $(u^n/n!)(x^k/k!)$ on both sides of (3.2) gives (3.3), or equivalently, (3.1).

In this paper we are primarily concerned with the characterization of r-umbralized polynomial sequences. As a corollary of Theorem 3.1 we have the following theorem.

THEOREM 3.2. *Let $\mathfrak{f}$ denote a given delta r-sequence. Then $p_n(x)$ is the umbralized sequence relative to $\mathfrak{f}$ if and only if*

$$\sum_{n=0}^{\infty} p_n(x)\frac{u^n}{n!} = \sum_{j=0}^{r-1} q_j(u)\exp_j(xh(u); r) \tag{3.4}$$

where

$$h(u) = (f_1(u)f_2(u)\cdots f_r(u))^{1/r}, \tag{3.5}$$

and for $0 \leqslant j \leqslant r-1$,

$$q_j(u) = f_0(u)f_1(u)\cdots f_j(u)/h(u)^j. \tag{3.6}$$

PROOF. By Theorem 3.1, we have that (3.2) holds. Rearranging the sum on the right-hand side of (3.2) according to the residues mod(r) of the exponents of x, we obtain

$$\begin{aligned} \sum_{k=0}^{\infty}\frac{x^k}{k!} f_0(u)\cdots f_k(u) &= f_0 \sum_{j=0}^{\infty}\frac{x^{rj}}{(rj)!}h^{rj} + \frac{f_0 f_1}{h}\sum_{j=0}^{\infty}\frac{x^{rj+1}}{(rj+1)!}h^{rj+1} \\ &\quad + \cdots + \frac{f_0 f_1\cdots f_{r-1}}{h^{r-1}}\sum_{j=0}^{\infty}\frac{x^{r(j+1)-1}}{(r(j+1)-1)!}h^{r(j+1)-1} \\ &= \sum_{j=0}^{r-1} q_j(u)\exp_j(xh(u); r), \end{aligned} \tag{3.7}$$

where the $q_j(u)$'s and $h(u)$ are as defined in (3.5) and (3.6) above.

The following two theorems give an elegant generating function characterization of r-umbralized polynomial sequences. Essentially we show that these sequences arise as sums of a related family of Sheffer sequences [9] evaluated at the values $\omega^j x$.

THEOREM 3.3. *Let* $\mathfrak{f}$ *denote a proper r-sequence with corresponding r-umbralized polynomial sequence* $\{p_n(x)\}_{n\geq 0}$, *and set* $h(u)=(f_1(u)f_2(u)\cdots f_r(u))^{1/r}$. *Then*

$$\sum_{n=0}^{\infty} p_n(x)\frac{u^n}{n!}=\sum_{j=0}^{r-1} B_j(u)\exp(\omega^j x h(u)), \tag{3.8}$$

where

$$B_j(0)=\delta_{0,j} \tag{3.9}$$

and

$$\sum_{j=0}^{r-1} B_j(u)=f_0(u). \tag{3.10}$$

PROOF. Set for $j=0,1,\ldots,r-1$,

$$B_j(u)=\frac{1}{r}\sum_{k=0}^{r-1}\omega^{-jk}q_k(u) \tag{3.11}$$

where for $k=0,1,\ldots,r-1$,

$$q_k(u)=f_0(u)f_1(u)\cdots f_k(u)/h(u)^k. \tag{3.12}$$

Since $\mathfrak{f}$ is a proper r-sequence, we have for all k that

$$q_k(0)=1. \tag{3.13}$$

Thus, (3.9) follows immediately from (3.11) and (3.13). Moreover, (3.10) holds since

$$\begin{aligned}\sum_{j=0}^{r-1} B_j(u)&=\sum_{k=0}^{r-1}\frac{1}{r}f_0f_1\cdots f_k\sum_{j=0}^{r-1}\omega^{-kj}\\ &=f_0(u).\end{aligned} \tag{3.14}$$

Finally, combining Proposition 2.1, Theorem 3.2 and (3.11) we have

$$\begin{aligned}\sum_{n=0}^{\infty} p_n(x)\frac{u^n}{n!}&=\sum_{k=0}^{r-1} q_k(u)\exp_k(xh(u);r)\\ &=\sum_{j=0}^{r-1}\exp(\omega^j xh(u))\sum_{k=0}^{r-1}\omega^{-kj}q_k(u)\\ &=\sum_{j=0}^{r-1}\exp(\omega^j xh(u))B_j(u),\end{aligned}$$

and our proof is complete.

Before we state the converse of Theorem 3.3 we need to make the following notational conventions. Let V denote the Vandermonde matrix on the powers of ω^{-1}; the (i,j)th entry of V is $(\omega^{-1})^{(i-1)(j-1)}$. Set $\Delta=\det(V)$, and let Δ_{ij} denote the signed (i,j)th cofactor of V. The converse of Theorem 3.3 is as follows.

THEOREM 3.4. *Let* $\{p_n(x)\}_{n\geq 0}$ *be a given polynomial sequence satisfying*

$$\sum_{n=0}^{\infty} p_n(x)\frac{u^n}{n!}=\sum_{j=0}^{r-1} B_j(u)\exp(\omega^j xh(u)) \tag{3.15}$$

such that

$$h(0)=0 \quad \textit{and} \quad h'(0)=1,$$

and

$$B_j(0)=\delta_{0,j}. \tag{3.17}$$

Set for notational convenience,

$$q_0(u)=q_r(u)=\sum_{j=0}^{r-1} B_j(u), \tag{3.18}$$

and for $1\leq k\leq r-1$,

$$q_k(u)=r\sum_{j=1}^{r}\frac{\Delta_{ij}}{\Delta}B_{j-1}(u).$$

Then $\{p_n(x)\}_{n\geq 0}$ *is the r-umbralized polynomial sequence relative to the proper r-sequence* $\mathfrak{f}$ *given by*

$$f_0(u)=q_0(u), \tag{3.20}$$

and for $1\leq k\leq r$,

$$f_k(u)=h(u)q_k(u)/q_{k-1}(u). \tag{3.21}$$

PROOF. Let us first show that Formulas (3.20) and (3.21) define a proper r-sequence. To this end note that Formulas (3.17), (3.18) and (3.20) guarantee that $f_0(0)=1$. In addition, (3.19) is equivalent to the assertion that

$$\begin{pmatrix} B_0\\ B_1\\ \vdots\\ B_{r-1}\end{pmatrix}=\frac{1}{r}V\begin{pmatrix} q_0\\ q_1\\ \vdots\\ q_{r-1}\end{pmatrix}. \tag{3.22}$$

Evaluating both sides of (3.22) at $u=0$ gives

$$\begin{pmatrix} 1\\ 0\\ \vdots\\ 0\end{pmatrix}=\frac{1}{r}V\begin{pmatrix} 1\\ q_1(0)\\ \vdots\\ q_{r-1}(0)\end{pmatrix}, \tag{3.23}$$

which clearly has the unique solution

$$q_1(0)=\cdots=q_{r-1}(0)=1. \tag{3.24}$$

Moreover, using (3.16), (3.21) and (3.24) we see that for all $k\geq 1$,

$$f_k(0)=0 \quad\text{and}\quad f_k'(0)=1.$$

Therefore, $\mathfrak{f}$ is a proper r-sequence.

Using Theorem 3.2, our proof will be complete once we have shown that under the present assumptions (3.4)–(3.6) hold. Since

$$\sum_{k=0}^{r-1} B_k(u)\exp(\omega^k xh(u))=\sum_{j=0}^{r-1} q_j(u)\exp_j(xh(u);r)$$

holds if and only if (3.22) holds, and (3.22) is equivalent to (3.19), we see that (3.4) is satisfied. Formulas (3.5) and (3.6) are easily seen to hold since (3.21) implies both that

$$f_1(u)\cdots f_r(u)=h(u)^r, \tag{3.25}$$

and for all $k=0, 1, \cdots, r-1$,

$$q_k(u)=f_0(u)\cdots f_k(u)/h(u)^k. \tag{3.26}$$

Whence, our proof is complete.

IV. On Umbralized and Generalized Sheffer Sequences

As previously remarked in Section I, polynomial sequences of binomial type and Sheffer sequences are special cases of r-umbralized polynomial sequences. Indeed, for $r>1$, r-umbralized polynomial sequences relative to normalized delta r-sequences generalize polynomials of binomial type, and r-umbralized polynomial sequences relative to arbitrary delta r-sequences generalize Sheffer sequences.

A different generalization of Sheffer sequences was given by Al-Salam and Verma [1]. A polynomial sequence is said to be an *r-generalized Sheffer sequence* if there exists a formal power series $J(u)$ such that

$$\operatorname{index} J(u)=r, \tag{4.1}$$

and for all $n\geq r$,

$$J(D)p_n(x)=p_{n-r}(x). \tag{4.2}$$

When $r=1$, (4.1) and (4.2) give a characterization of Sheffer sequences. Al-Salam and Verma show that the polynomial sequence $\{p_n(x)\}_{n\geq 0}$ is an r-generalized Sheffer sequence if and only if there exists a compositionally invertible formal power series $h(u)$ such that

$$\sum_{n=0}^{\infty} p_n(x)\frac{u^n}{n!}=\sum_{j=0}^{r-1} A_j(u)\exp(xh(\omega^j u)), \tag{4.3}$$

where for some $0\leq j\leq r-1$, $A_j(0)\neq 0$. Moreover, it turns out that $h(u)$ is the compositional inverse of the formal power series $(J(u))^{1/r}$.

Comparing formulas (3.8) and (4.3) we see that essentially, a polynomial sequence is both an r-umbralized polynomial sequence and an r-generalized Sheffer sequence whenever for all $j=0, 1, \ldots, r-1$,

$$h(\omega^j u)=\omega^j h(u), \tag{4.4}$$

or equivalently whenever there exists a formal power series $g(u)$ with zero index such that

$$h(u)=ug(u^r). \tag{4.5}$$

Necessity in (4.5) is seen as follows. If (4.4) holds for all $j=0, 1, 2, \ldots, r-1$, then the case $j=1$ implies that for all $n\geq 1$ we have

$$\omega^n(h(u)|_{u^n})=\omega(h(u)|_{u^n}). \tag{4.6}$$

Since ω is a primitive rth root of unity we see that the only terms u^n in $h(u)$ that can have non-zero coefficients are those where $n=rk+1$ for some $k\geq 0$. Therefore, setting the coefficient of u^k in $g(u)$ equal to the coefficient of u^{rk+1} in $h(u)$, we obtain (4.5).

V. Convolution Formulas

Let $\mathbf{f}=\{f_0, f_1, \ldots, f_r\}$ denote a given proper r-sequence. Let us set

$$\mathbf{f}^{(0)}=\mathbf{f},$$

and for $1\leq j\leq r-1$,

$$\mathbf{f}^{(j)}=\{f_0, f_{j+1}, \ldots, f_r, f_1, \ldots, f_j\}. \tag{5.1}$$

In addition, for all $0 \leqslant j \leqslant r-1$ and all $n \geqslant 0$, we shall set

$$p_n^{(j)}(x) = U_{\mathfrak{t}^{(j)}} x^n, \tag{5.2}$$

and

$$G_j(x, u) = \sum_{n=0}^{\infty} p_n^{(j)}(x) \frac{u^n}{n!}. \tag{5.3}$$

Polynomial sequences of binomial type satisfy a particularly simple convolution identity (see (1.3)). Since r-umbralized polynomial sequences generalize binomial type sequences, we shall pursue in this section an investigation of the convolution formulas for the polynomial sequences $\{p_n^{(j)}(x+y)\}_{n \geqslant 0}$. Some of the most elegant convolution formulas in this setting occur when we consider the polynomial sequences $\{s_n^{(j)}(x)\}$ defined by setting for all $n \geqslant 0$ and $0 \leqslant j \leqslant r-1$,

$$s_n^{(j)}(x) = \sum_{k=0}^{r-1} \omega^{-jk} p_n^{(k)}(x), \tag{5.4}$$

or equivalently

$$\begin{pmatrix} s_n^{(0)}(x) \\ s_n^{(1)}(x) \\ \vdots \\ s_n^{(r-1)}(x) \end{pmatrix} = V \begin{pmatrix} p_n^{(0)}(x) \\ p_n^{(1)}(x) \\ \vdots \\ p_n^{(r-1)}(x) \end{pmatrix}.$$

Here we shall present the convolution identities for the sequences $\{s_n^{(j)}(x)\}_{n \geqslant 0}$.

We denote the generating functions for these sequences by $H_j(x, u)$, that is, for $0 \leqslant j \leqslant r-1$,

$$\begin{aligned} H_j(x, u) &= \sum_{n=0}^{\infty} s_n^{(j)}(x) \frac{u^n}{n!} \\ &= \sum_{k=0}^{r-1} \omega^{-jk} G_j(x, u). \end{aligned} \tag{5.5}$$

For notational convenience, we extend the range of the parameter j in (5.4) and (5.5) to all integers by requiring that for all $n \geqslant 0$ and all j,

$$s_n^{(j+r)}(x) = s_n^{(j)}(x) \tag{5.6}$$

and

$$H_{j+r}(x, u) = H_j(x, u). \tag{5.7}$$

THEOREM 5.1. *For each $j = 0, 1, \ldots, r-1$, the following generating function identity holds.*

$$G_j(x+y, u) = \sum_{\nu=0}^{r-1} \omega^{\nu j} G_j(\omega^{\nu} x, u) H_{\nu}(y, u). \tag{5.8}$$

PROOF. Let us first consider the case $j = 0$. By Theorem 3.3,

$$G_0(x+y, u) = \sum_{j=0}^{r-1} q_j(u) \exp_j((x+y)h(u); r), \tag{5.9}$$

where $q_j(u)$ is as defined in (3.12). Expanding the right-hand side of (5.9) via the addition formulas of Proposition 2.2 gives

$$\begin{aligned} G_0(x+y,u) &= \sum_{j=0}^{r-1} q_j(u) \sum_{\nu=0}^{r-1} \exp_\nu(x;r)\exp_{r+j-\nu}(y;r) \\ &= \sum_{\nu=0}^{r-1} q_\nu(u)\exp_\nu(x;r) \sum_{j=0}^{r-1} (q_j(u)/q_\nu(u))\exp_{r+j-\nu}(y;r). \end{aligned} \tag{5.10}$$

Moreover, due to the cyclicity of the r-sequences $\mathbf{f}^{(j)}$ and Theorem 3.3, it is not difficult to see that

$$\sum_{j=0}^{r-1} (q_j(u)/q_\nu(u))\exp_{r+j-\nu}(y;r) = G_\nu(y;u). \tag{5.11}$$

Therefore, combining (5.10) and (5.11) we have that

$$G_0(x+y,u) = \sum_{\nu=0}^{r-1} q_\nu(u)\exp_\nu(x;r)G_\nu(y,u). \tag{5.12}$$

Using Theorem 3.3 once again we see that

$$q_\nu(u)\exp_\nu(x;r) = \sum_{j=0}^{r-1} G_0(\omega^j x,u)\omega^{-\nu j} \tag{5.13}$$

and substituting (5.13) into (5.12) gives the desired

$$\begin{aligned} G_0(x+y,u) &= \sum_{j=0}^{r-1} G_0(\omega^j x,u) \sum_{\nu=0}^{r-1} \omega^{-\nu j} G_\nu(y,u) \\ &= \sum_{j=0}^{r-1} G_0(\omega^j x,u)H_j(y,u). \end{aligned} \tag{5.14}$$

To obtain the result for $j=1,2,\ldots,r-1$, we observe that if we replace $G_0(x,u)$ by $G_j(x,u)$ and use precisely the same arguments we have

$$G_j(x+y,u) = \sum_{\nu=0}^{r-1} G_j(\omega^\nu x,u) \sum_{\mu=0}^{r-1} \omega^{-\nu\mu} G_{\mu+j}(y,u). \tag{5.15}$$

Moreover it is easily seen that

$$\sum_{\mu=0}^{r-1} \omega^{-\nu\mu} G_{\mu+j}(y,u) = \omega^{\nu j} H_\nu(y,u). \tag{5.16}$$

Combining (5.15) and (5.16) gives the desired (5.8) and our proof is complete.

Our convolution formulas follow from the following theorem.

THEOREM 5.2. *Let $H_j(x,u)$ be as defined in* (5.5). *Then, for all* $0 \leq j \leq r-1$,

$$H_j(x+y,u) = \sum_{\nu=0}^{r-1} H_{r+j-\nu}(\omega^\nu x,u)H_\nu(y,u). \tag{5.17}$$

PROOF. Using Theorem 5.1 we have

$$\begin{aligned} H_j(x+y,u) &= \sum_{\mu=0}^{r-1} G_\mu(x+y,u)\omega^{-\mu j} \\ &= \sum_{\nu=0}^{r-1} H_\nu(y,u) \sum_{\mu=0}^{r-1} \omega^{-\mu(j-\nu)} G_\mu(\omega^\nu x,u) \\ &= \sum_{\nu=0}^{r-1} H_\nu(y,u)H_{r+j-\nu}(\omega^\nu x,u), \end{aligned}$$

as asserted.

As a corollary of Theorem 5.2 we have the following theorem.

THEOREM 5.3. *Let* $\{s_n^{(j)}(x)\}_{n\geq 0}$ *be as defined in* (5.4). *Then, for each* $j=0,1,\ldots,r-1$, *the following convolution identity holds*:

$$s_n^{(j)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}\sum_{\nu=0}^{r-1}s_k^{(r+j-\nu)}(\omega^{\nu}x)s_{n-k}^{(\nu)}(y). \tag{5.18}$$

PROOF. Formula (5.18) follows immediately upon equating the coefficients of u^n on both sides of (5.17).

VI. EQUIVALENCE CLASSES OF UMBRALIZED SEQUENCES

Two polynomial sequences $\{p_n(x)\}_{n\geq 0}$ and $\{q_n(x)\}_{n\geq 0}$ are said to be *compositionally equivalent* [5], or *C-equivalent* for short, if there exists a binomial type sequence $\{\gamma_n(x)\}_{n\geq 0}$ such that the umbral composition of $\{p_n(x)\}$ with $\{\gamma_n(x)\}$ gives $\{q_n(x)\}$; that is, if for all $n\geq 0$,

$$p_n(x)=\sum_{k=0}^{n}p_{n,k}x^k,$$

then

$$p_n(\boldsymbol{\gamma})=\sum_{k=0}^{n}p_{n,k}\gamma_k(x)=q_n(x). \tag{6.1}$$

It is well known [9] that every polynomial sequence of binomial type is C-equivalent to every other polynomial sequence of binomial type. Since $\{x^n\}_{n\geq 0}$ is a binomial type sequence, there exists a binomial type sequence $\{\alpha_n(x)\}_{n\geq 0}$ such that, for all $n\geq 0$,

$$\gamma_n(\boldsymbol{\alpha})=x^n.$$

Therefore, umbrally composing both sides of (6.1) with $\{\alpha_n(x)\}_{n\geq 0}$ gives, for all $n\geq 0$,

$$p_n(x)=q_n(\boldsymbol{\alpha}),$$

and thus we see that C-equivalence is indeed an equivalence relation. In this section we shall study the C-equivalence of r-umbralized polynomial sequences.

THEOREM 6.1. *Let* $\{p_n(x)\}_{n\geq 0}$ *be the r-umbralized polynomial sequence relative to the normalized r-sequence* $\mathbf{f}=\{1,f_1(u),\ldots,f_r(u)\}$, $\{\gamma_n(x)\}$ *be the binomial type sequence relative to a given, compositionally invertible series* $a(u)$, *and set*

$$p_n(\boldsymbol{\gamma})=\sum_{k=0}^{n}p_{n,k}\gamma_k(x). \tag{6.2}$$

Then the sequence $\{p_n(\boldsymbol{\gamma})\}_{n\geq 0}$ *is the r-umbralized polynomial sequence relative to the normalized r-sequence* $\mathbf{f}(a)=\{1,f_1(a(u)),\ldots,f_r(a(u))\}$. *Conversely, if* $\{q_n(x)\}_{n\geq 0}$ *is the r-umbralized polynomial sequence relative to* $\{1,f_1(b(u)),\ldots,f_r(b(u))\}$ *and* $b(u)$ *is compositionally invertible, then* $\{q_n(x)\}_{n\geq 0}$ *is C-equivalent to* $\{p_n(x)\}_{n\geq 0}$.

PROOF. It is easy to see that, for all $n\geq 0$,

$$p_n(\boldsymbol{\gamma})=U_{\mathbf{f}}U_a x^n. \tag{6.3}$$

In addition,

$$
\begin{aligned}
U_{\mathbf{f}}U_a &= \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu!} L_0 f_1(D)\cdots f_\nu(D) \sum_{k=0}^{\infty} \frac{x^k}{k!} L_0 a(D)^k \\
&= \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu!} L_0 \sum_{k=0}^{\infty} (f_1(u)\cdots f_\nu(u)|_{u^k}) a(D)^k \\
&= \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu!} L_0 f_1(a(D))\cdots f_\nu(a(D)) \\
&= U_{\mathbf{f}(a)}.
\end{aligned} \tag{6.4}
$$

Combining (6.3) and (6.4) gives

$$p_n(\boldsymbol{\gamma}) = U_{\mathbf{f}(a)} x^n,$$

as asserted. The converse follows from (6.4) since if $\{\beta_n(x)\}_{n\geq 0}$ is the binomial type sequence relative to $b(u)$, then

$$U_{\mathbf{f}(b)} = U_{\mathbf{f}} U_b$$

implies that, for all $n \geq 0$,

$$q_n(x) = p_n(\boldsymbol{\beta}),$$

or that $\{q_n(x)\}_{n\geq 0}$ is C-equivalent to $\{p_n(x)\}_{n\geq 0}$.

Let B_r denote the class of all r-umbralized polynomial sequences that arise from *normalized* r-sequences. In light of Theorem 6.1, we see that the C-equivalence class of any polynomial sequence in B_r is contained in B_r.

THEOREM 6.2. *The C-equivalence classes of B_r are in bijection with the collection of all $(r-1)$-tuples of compositionally invertible formal power series.*

PROOF. Let A_r denote the class of all r-umbralized polynomial sequences relative to normalized r-sequences of the form

$$\mathbf{f} = \{1, u, f_2(u), \ldots, f_r(u)\}.$$

We claim that A_r forms a complete set of distinct representatives for the C-equivalence classes of B_r. To see this note that if $\{p_n(x)\}_{n\geq 0}$ and $\{q_n(x)\}_{n\geq 0}$ are sequences in A_r, and arise respectively from $\mathbf{f}$ and $\mathbf{g}$, then using Theorem 6.1, $\{p_n(x)\}_{n\geq 0}$ is C-equivalent to $\{q_n(x)\}_{n\geq 0}$ if and only if, for all $j = 1, 2, \ldots, r$.

$$f_j(u) = g_j(u).$$

Therefore, no two distinct sequences in A_r lie in the same C-equivalence class. Now let $\{p_n(x)\}_{n\geq 0}$ be in B_r, and suppose $\{p_n(x)\}_{n\geq 0}$ is relative to $\mathbf{f} = \{1, f_1, \ldots, f_r\}$. Let $a(u)$ denote the compositional inverse of $f_1(u)$, and let $\{\gamma_n(x)\}_{n\geq 0}$ be the binomial type sequence relative to $a(u)$. Then $\{p_n(\boldsymbol{\gamma})\}_{n\geq 0}$ is C-equivalent to $\{p_n(x)\}_{n\geq 0}$ and it is also an element of A_r. Thus A_r is seen to be a complete set of distinct representatives for the C-equivalence classes of B_r, and our proof is complete.

In this paper we have developed the generating function and convolution characterizations of r-umbralized polynomial sequences. For polynomial sequences of binomial type, there is a third characterization, namely the delta operator characterization which states that $\{p_n(x)\}_{n\geq 0}$ is of binomial type if and only if, for all $n \geq 0$,

$$L_0 p_n(x) = L_0 x^n \tag{6.5}$$

and there exists a compositionally invertible formal power series $F(u)$ such that

$$F(D)p_n(x) = np_{n-1}(x). \tag{6.6}$$

For binomial type sequences, the equivalence of the generating function and delta operator characterizations involves the Lagrange inversion formula. Several generalizations of Lagrange inversion have been recently proposed [4, 6, 10], and further investigation along these lines may lead to a delta operator characterization of r-umbralized polynomial sequences.

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S. A. JONI
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, U.S.A.